UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH7501**
ASSESSMENT PATTERN	:	**MATH7501A**
MODULE NAME	:	**Probability and Statistics**
DATE	:	**01-May-09**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2008/09-MATH7501A-001-EXAM-153

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is *permitted in this examination.*

New Cambridge Statistical Tables are provided.

1. (a) Define each of the following terms, as used in probability theory:
   (i) Sample space.
   (ii) Event.
   (iii) Random variable.

   (b) Consider an event space $\mathcal{F}$. State the three axioms of probability for a probability function $P(\cdot)$ defined on $\mathcal{F}$.

   (c) What is meant by saying that events A, B and C are (i) mutually disjoint (ii) mutually independent?

   (d) Suppose $P(A \cap B) > 0$. Show that $P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$.

   (e) Suppose that $P(A) > 0$ and $P(B) > 0$. Show that if $P(A|B) > P(A)$ then $P(B|A) > P(B)$.

2. In a collection of seven, apparently identical, coins there are two fair coins. Of the other five coins, two always land showing a head, while the other three always land showing a tail. A coin is chosen at random from the seven, and tossed.

   (a) What is the probability that the coin will land with the head showing? If X is a random variable that takes the value 1 if a head is shown and 0 otherwise, what is $E(X)$?

   (b) Suppose now that the coin does land with the head showing. What is the probability that it is one of the fair coins?

   (c) If the same coin is tossed a second time, and given that the first toss is a head, what is the probability that another head will be obtained?

   (d) Given that a second head is obtained, what is the probability that this coin is one of the fair coins?

3. The lifetime of a certain type of light bulb is exponentially distributed with a mean of 1000 hours; lifetimes of different bulbs are independent.

   (a) What proportion of bulbs have lifetimes in excess of 100 hours?

   (b) Bulbs are packaged in boxes of 2. The manufacturer guarantees that in each box both the 2 bulbs will have lifetimes in excess of 100 hours. What percentage of boxes will meet the guarantee?

   (c) Suppose the bulbs are used in the following way: one bulb is used at a time, and is replaced by a new one at the time it burns out.

      (i) Explain how a Poisson process describes the times at which light bulbs burn out.

      (ii) Show that the probability of using up two full boxes of 4 bulbs within t hours is $1 - \sum_{k=0}^{3} \frac{e^{-t/1000}(t/1000)^k}{k!}$ for any $t > 0$.

   (d) Let Z_2 be the time until 2 boxes of 4 bulbs are used up. Find the probability density function of Z_2 and the state its mean.

4. Let $Z_1, \ldots, Z_n$ be independent Bernoulli random variables each with parameter p. Let $S_n = \sum_{i=1}^{n} Z_i$.

   (a) Find the probability distribution of S_n, and hence state its mean and variance.

   (b) If $n = 100$ and $p = 0.04$, find the exact probability that $S_n \leqslant 2$.

   (c) Find an approximation to the above probability

      (i) using the Poisson approximation to the Binomial,

      (ii) using the Normal approximation to the Binomial.

   (d) Which of the above approximations is better? Why?

5. Suppose that $X_1, \ldots, X_n$ are independent Exponential random variables each with parameter $1/\alpha$. Let $Z = X_1 + \ldots + X_n$ and let $Y = \min\{X_1, \ldots, X_n\}$.

   (a) Determine $E(Z)$ and $Var(Z)$.

   (b) Derive an expression for $P(Y > y)$, $y > 0$, in terms of α and use it to identify that $Y \sim Exp(n/\alpha)$. (Note that the event $\{Y > y\}$ is the same as the event $\cap_{i=1}^{n}\{X_i > y\}$.) State the mean and variance of Y.

   (c) Use the results in (a) and (b) to find constants A and B such that AZ and BY are unbiased estimators of α.

   (d) Compare the variances of the two estimators obtained in (c). Which estimator would you prefer, and why?

6. For a certain game, individual game scores are normally distributed. Two players played 10 games each, and recorded their scores on each game. For player A, the average score is 375 and the sample variance is 17312. For player B, the average score is 360 and the sample variance is 13208.

   (a) Test, at the 5% level the hypothesis that the variances of the two players' scores are the same assuming that the true means are unknown.

   (b) Test, at the 5% level the hypothesis that the mean scores of the two players are the same under the assumption that the variances are the same.

   (c) Assuming that the variances are the same, construct a 95% confidence interval for the difference of the mean scores of the two players.